2963. [2004 : 367, 370] *Proposed by Mihály Bencze, Brasov, Romania.*

Let ABC be any acute-angled triangle. Let r and R be the inradius and circumradius, respectively, and let s be the semiperimeter; that is, $s = \frac{1}{2}(a+b+c)$. Let m_a be the length of the median from A to BC, and let w_a be the length of the internal bisector of $\angle A$ from A to the side BC. We define m_b, m_c, w_b and w_c similarly. Prove that

(a) $\dfrac{3s^2 - r^2 - 4Rr}{8sRr} \le \displaystyle\sum_{\text{cyclic}} \frac{m_a}{aw_a} \le \frac{s^2 - r^2 - 4Rr}{7sRr}$;

(b) $\dfrac{3}{4} \le \displaystyle\sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} \le \frac{4R+r}{4R}$.

Solution to part (a) by Arkady Alt, San Jose, CA, USA.

First we note that the right inequality is incorrect. For example, if ABC is equilateral, then

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} = \frac{3}{a} > \frac{6}{7a} = \frac{s^2 - r^2 - 4Rr}{7sRr}.$$

We will instead prove that

$$\frac{3s^2 - r^2 - 4Rr}{8sRr} \le \sum_{\text{cyclic}} \frac{m_a}{aw_a} \le \frac{s^2 - r^2 - 4Rr}{2sRr}.$$

We use the following well-known identities:

$$\begin{aligned} 4m_a^2 &= 2(b^2+c^2) - a^2, \\ w_a^2 &= \frac{bc\big((b+c)^2 - a^2\big)}{(b+c)^2}, \\ abc &= 4sRr, \\ a^2+b^2+c^2 &= 2s^2 - 2r^2 - 8Rr, \\ ab+bc+ca &= s^2 + r^2 + 4Rr. \end{aligned}$$

From the first two identities above, we get

$$\frac{m_a^2}{w_a^2} = \frac{(b+c)^2}{4bc} \cdot \frac{2(b^2+c^2) - a^2}{(b+c)^2 - a^2}. \tag{1}$$

Now we observe that

$$2bc \le (b+c)^2 - a^2 \le 4bc. \tag{2}$$

The left inequality is true because it is equivalent to $b^2 + c^2 \ge a^2$, which is true for any acute triangle, and the right inequality is true because it is equivalent to $|b-c| \le a$, which is true for any triangle. From (2), we get

$$\frac{(b-c)^2}{4bc}+1 \;\le\; \frac{(b-c)^2}{(b+c)^2-a^2}+1 \;\le\; \frac{(b-c)^2}{2bc}+1\,;$$

that is,

$$\frac{(b+c)^2}{4bc} \;\le\; \frac{2(b^2+c^2)-a^2}{(b+c)^2-a^2} \;\le\; \frac{b^2+c^2}{2bc}\,.$$

Recalling (1), we get

$$\frac{(b+c)^4}{16b^2c^2} \;\le\; \frac{m_a^2}{w_a^2} \;\le\; \frac{(b+c)^2(b^2+c^2)}{8b^2c^2}\,.$$

Now, using the easy-to-prove inequality $(b+c)^2 \le 2(b^2+c^2)$, we obtain

$$\frac{(b+c)^4}{16b^2c^2} \;\le\; \frac{m_a^2}{w_a^2} \;\le\; \frac{(b^2+c^2)^2}{4b^2c^2}\,.$$

Taking square roots throughout and dividing by a gives

$$\frac{(b+c)^2}{4abc} \;\le\; \frac{m_a}{aw_a} \;\le\; \frac{b^2+c^2}{2abc}\,,$$

where equality occurs if and only if $b=c$.

Using similar inequalities for $\frac{m_b}{bw_b}$ and $\frac{m_c}{cw_c}$, we obtain

$$\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;\le\; \sum_{\text{cyclic}} \frac{b^2+c^2}{2abc} \;=\; \frac{a^2+b^2+c^2}{abc} \;=\; \frac{s^2-r^2-4Rr}{2sRr}$$

and

$$\begin{aligned}\sum_{\text{cyclic}} \frac{m_a}{aw_a} \;&\ge\; \sum_{\text{cyclic}} \frac{(b+c)^2}{4abc} \;=\; \frac{1}{2abc}\left(\sum_{\text{cyclic}} a^2 + \sum_{\text{cyclic}} bc\right)\\ &=\; \frac{1}{8sRr}(2s^2-2r^2-8Rr+s^2+r^2+4Rr)\\ &=\; \frac{3s^2-r^2-4Rr}{8sRr}\,,\end{aligned}$$

as claimed. Equality occurs in both inequalities if and only if $a=b=c$.

Solution to part (b) by Michel Bataille, Rouen, France.

We prove that

$$\frac{3}{4} \;<\; \sum_{\text{cyclic}} \frac{m_a^2}{b^2+c^2} \;\le\; \frac{4R+r}{4R}\,.$$

Since $4m_a^2 = 2(b^2+c^2) - a^2$, it is easily seen that our inequality is equivalent to

$$2 - \frac{r}{R} \;\le\; \sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} \;<\; 3\,.$$

Since the triangle is acute, the cosines of all angles are positive. Using the Cosine Law, we obtain

$$\sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} \;=\; \sum_{\text{cyclic}} \frac{b^2+c^2-2bc\cos A}{b^2+c^2} \;=\; 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2+c^2} \;<\; 3\,.$$

On the other hand, using the well-known identity

$$\cos A + \cos B + \cos C \;=\; 1 + \frac{r}{R}$$

and the easy inequality $2bc \le b^2+c^2$, we obtain

$$\begin{aligned} \sum_{\text{cyclic}} \frac{a^2}{b^2+c^2} &= 3 - \sum_{\text{cyclic}} \frac{2bc\cos A}{b^2+c^2} \;\ge\; 3 - \sum_{\text{cyclic}} \cos A \\ &= 3 - \left(1 + \frac{r}{R}\right) \;=\; 2 - \frac{r}{R}\,. \end{aligned}$$

Equality holds if and only if $a = b = c$.

Also solved by ARKADY ALT, San Jose, CA, USA (part (b)); ŠEFKET ARSLANAGIĆ, University of Sarajevo, Sarajevo, Bosnia and Herzegovina (part (b)); MICHEL BATAILLE, Rouen, France (part (a)); JOHN G. HEUVER, Grande Prairie, AB; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria (part (b)); VEDULA N. MURTY, Dover, PA, USA; PANOS E. TSAOUSSOGLOU, Athens, Greece; and the proposer.

Janous believes that the lower bound of $3/4$ in inequality (b) can be increased to 1, but he does not have a proof. We encourage our readers to try to find a bound better than $3/4$.

The editors apologize for the typo in the right side of the inequality of part (a). The proposer's version was the correct one (found also by Alt and Bataille). Several other solvers either gave a counterexample or suggested a correct version and solved it.